LOCAL GROMOV–WITTEN INVARIANTS OF CUBIC SURFACES VIA NEF TORIC DEGENERATION

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Abstract. We compute local Gromov–Witten invariants of cubic surfaces at all genera. We use a deformation of a cubic surface to a nef toric surface and the deformation invariance of Gromov–Witten invariants.

1. Introduction

A del Pezzo surface S_d of degree d $(1 \leq d \leq 9)$ [1] is a smooth surface[2] whose anticanonical divisor $-K_{S_d}$ is ample and $(-K_{S_d})^2 = d$. For a smooth projective surface X, the local Gromov–Witten (GW) invariant is a rational number defined by the integral of a certain class, which is determined by the canonical divisor K_X, on the moduli stack of stable maps to X [4, 16]. Local GW invariants of del Pezzo surfaces have been intensively studied in physics in relation to the non-critical string by various methods: mirror symmetry, Seiberg–Witten curve technique and so on (see e.g. [22]). In the case of toric del Pezzo surfaces (i.e. $6 \leq d \leq 9$), a powerful method based on the duality to the Chern–Simons theory enables us to write down an explicit formula for the generating function at all genera [5, 6, 1, 14]. The formula was proved in [31] based on the virtual localization [21, 11] together with a formula for Hodge integrals [24]. In a recent interesting work [8], Diaconescu and Florea proposed a closed formula for the generating function of nontoric del Pezzo surfaces S_i $(1 \leq i \leq 5)$ for all genera by using the conjectural ruled vertex formalism [7].

Our modest goal is to obtain a formula for the generating function of local GW invariants of S_3 at all genera. S_3 is isomorphic to $\mathbb{P}^2$ blown-up at 6 points in a general position and it is also realized as a smooth cubic surface in $\mathbb{P}^3$. It is not toric but have a (unique) smooth nef toric degeneration S_3^0 (a smooth toric surface with the nef anticanonical divisor which is deformation equivalent to S_3). A main idea is to use the deformation invariance of local GW invariants as in [8, 30] and reduce the computation to those of S_3^0 where we can apply the virtual localization. Here we remark that our results are limited to S_k $(k = 3, 4, 5)$ since S_1 and S_2 do not admit nef toric degenerations.

The results of this paper are as follows. We first prove that in the case of a smooth projective surface with the nef anticanonical divisor, local GW invariants are equal to ordinary GW invariants of a projective bundle compactification of the total space of the canonical line bundle (Proposition 2.2). Our proof is based on the virtual localization with respect to the $\mathbb{C}^*$-action in the fiber direction. Then the deformation invariance of the latter [23, 29]

2000 *Mathematics Subject Classification.* Primary 14N35, 53D45; Secondary 14J26, 14J32.

[1]In physics literatures, S_d is usually denoted by dP_{9-d} or B_{9-d}. Here we follow the notation used in [26, §0]. A brief account of the classification of del Pezzo surfaces can be found there.

[2]In this article, a surface means an algebraic surface over $\mathbb{C}$.

implies that of the former (Proposition 2.4). Next we introduce the toric surface S_3^0 and show that it is the nef toric degeneration of S_3 (Proposition 4.1). Then we derive a formula for the generating function of local GW invariants of S_3^0 by the virtual localization (Lemma 5.1). Finally we obtain a formula for the generating function of local GW invariants of S_3 via those of S_3^0 by the deformation invariance (Theorem 5.2).

The organization of the paper is as follows. In Section 2, we give a definition of local GW invariants and show the deformation invariance. In Section 3, we summarize necessary facts about cubic surfaces S_3. In Section 4, we introduce the toric surface S_3^0. For completeness, a proof of the deformation equivalence of S_3 and S_3^0 is included in Appendices A and B. In Section 5, we give formulas for the generating functions of local GW invariants of S_3^0 and S_3. We have computed the formula explicitly for $\beta \in H_2(S_3,\mathbb{Z})$ such that $-K_{S_3}.\beta \le 6$. The results are listed in Section 6 and Appendix C.

ACKNOWLEDGMENTS

Y.K. thanks Shinobu Hosono for valuable discussions, comments and helpful advice on the numerical computation. She also thanks Florin Ambro for helping her with the proof of Lemma B.1. S.M. is grateful to Hiroaki Kanno for many advices and encouragements. He is also grateful to Yuichi Nohara for discussions on the proof of Proposition 4.1. Both authors would like to thank Bogdan Florea and Kazushi Ueda for useful comments on the draft. The research of Y.K. is supported by JSPS Research Fellowships for Young Scientists.

The authors thanks Hiroshi Iritani, Hiraku Nakajima, Kaoru Ono and Kōta Yoshioka for pointing out an error in our original definition of the local GW invariants.

2. Deformation invariance of local GW invariants

In this article, we call a smooth projective surface X whose anticanonical divisor $-K_X$ is nef (i.e. $-K_X.[C] \ge 0$ for all curves $C \subset X$) a nef surface.

Let X be a nef surface and K_X its canonical divisor. For $\beta \in H_2(X,\mathbb{Z})$ and $g \in \mathbb{Z}_{\ge 0}$, let $\bar{M}_{g,0}(X,\beta)$ (resp. $\bar{M}_{g,1}(X,\beta)$) be the moduli stack of stable maps to X of genus g without marked point (resp. with one marked point) and with the second homology class β. Let $\pi : \bar{M}_{g,1}(X,\beta) \to \bar{M}_{g,0}(X,\beta)$ be the forgetful map of the marked point and $\mu : \bar{M}_{g,1}(X,\beta) \to X$ be the evaluation at the marked point.

Definition 2.1. For $g \in \mathbb{Z}_{\ge 0}$ and $\beta \in H_2(X,\mathbb{Z})$ such that $\int_\beta c_1(K_X) < 0$, the local Gromov–Witten invariant $N_{g,\beta}(K_X)$ of X with genus g and the second homology class β is

$$N_{g,\beta}(K_X) = \int_{[\bar{M}_{g,0}(X,\beta)]^{vir}} c_{top}(R^1\pi_*\mu^* K_X),$$

where c_{top} denotes the top Chern class which is of degree $(1-g)(\dim X - 3) - \int_\beta c_1(K_X)$. (This is equal to the virtual dimension of $\bar{M}_{g,0}(X,\beta)$.)[3]

[3]The condition $\int_\beta c_1(K_X) < 0$ and the nef condition on X imply $H^0(C, f^*K_X) = 0$ for $(f,C) \in \bar{M}_{g,0}(X,\beta)$.

Let $\mathbb{P}(K_X \oplus \mathcal{O}_X)$ be the projectivization of the total space of the vector bundle $K_X \oplus \mathcal{O}_X$ (here the canonical divisor K_X and the structure sheaf $\mathcal{O}_X$ are regarded as line bundles). This is a $\mathbb{P}^1$-bundle over X. Let $\iota : X \hookrightarrow \mathbb{P}(K_X \oplus \mathcal{O}_X)$ be the inclusion as the zero section of $K_X \subset \mathbb{P}(K_X \oplus \mathcal{O}_X)$. We define the (ordinary) GW invariant $N_{g,\iota_*\beta}(\mathbb{P}(K_X \oplus \mathcal{O}_X))$ of $\mathbb{P}(K_X \oplus \mathcal{O}_X)$ of genus g and the second homology class $\iota_*\beta$ by

$$N_{g,\iota_*\beta}(\mathbb{P}(K_X \oplus \mathcal{O}_X)) = \int_{[\bar{M}_{g,0}(\mathbb{P}(K_X \oplus \mathcal{O}_X),\iota_*\beta)]^{vir}} 1 \ .$$

We note that the deformation invariance is established for this ordinary GW invariant [23, 29].

Proposition 2.2. *Let X be a nef surface, $\iota : X \hookrightarrow \mathbb{P}(K_X \oplus \mathcal{O}_X)$ be the inclusion as the zero section of K_X. For $g \in \mathbb{Z}_{\geq 0}$ and $\beta \in H_2(X,\mathbb{Z})$ such that $\int_\beta c_1(K_X) < 0$,*

$$N_{g,\beta}(K_X) = N_{g,\iota_*\beta}(\mathbb{P}(K_X \oplus \mathcal{O}_X)).$$

Consider the natural $\mathbb{C}^*$ action on $\mathbb{P}(K_X \oplus \mathcal{O}_X)$ as the scalar multiplication in the $\mathbb{P}^1$-fiber direction. The action induces an action on $\bar{M}_{g,0}(\mathbb{P}(K_X \oplus \mathcal{O}_X), \iota_*\beta)$ by moving the image curves of stable maps. First we show the following lemma.

Lemma 2.3. *Let X be a nef surface, $\iota : X \hookrightarrow \mathbb{P}(K_X \oplus \mathcal{O}_X)$ be the inclusion as the zero section of K_X. Let $\beta \in H_2(X,\mathbb{Z})$ be a class satisfying $\int_\beta c_1(K_X) < 0$. If a stable map $(f,C) \in \bar{M}_{g,0}(\mathbb{P}(K_X \oplus \mathcal{O}_X), \iota_*\beta)$, where C is a connected curve of genus g and $f : C \to X$ a morphism such that $[f(C)] = \iota_*\beta$, is fixed by the $\mathbb{C}^*$-action, then the image $f(C)$ is contained in the zero section $\iota(X)$.*

Proof. Denote the $\mathbb{P}^1$-fibration $\mathbb{P}(K_X \oplus \mathcal{O}_X) \to X$ by p, and let $P = [p^{-1}(a)] \in H_2(\mathbb{P}(K_X \oplus \mathcal{O}_X), \mathbb{Z})$ be the class of the fiber $\mathbb{P}^1$ where $a \in X$ is any point. Let $\iota^\infty : X \hookrightarrow \mathbb{P}(K_X \oplus \mathcal{O}_X)$ be the inclusion as the zero section of $\mathcal{O}_X$ (the section at the infinity of the $\mathbb{P}^1$-bundle compactification of K_X). Note that for any $\alpha \in H_2(X,\mathbb{Z})$, we have

$$\iota^\infty_*\alpha = \iota_*\alpha - \left(\int_\alpha c_1(K_X)\right) P \ . \tag{2.1}$$

Let $\gamma \in H_2(\mathbb{P}(K_X \oplus \mathcal{O}_X), \mathbb{Z})$. If a stable map $(f,C) \in \bar{M}_{g,0}(\mathbb{P}(K_X \oplus \mathcal{O}_X), \gamma)$ is fixed by the $\mathbb{C}^*$-action, then the image of an irreducible component C_i of C must be either one of these: (i) $f(C_i) \subset \iota(X)$, (ii) $f(C_i) \subset \iota^\infty(X)$ or (iii) $f(C_i) = p^{-1}(a_i)$ $(a_i \in X)$ and $C_i \cong \mathbb{P}^1$. So assume that irreducible components $C_1, \ldots, C_k$ of C are of type (i) with $[f(C_i)] = \beta_i \in H_2(X,\mathbb{Z})$, $C_{k+1}, \ldots, C_r$ are of type (ii) with $[f(C_i)] = \beta_i \in H_2(X,\mathbb{Z})$, and that $C_{r+1}, \ldots, C_s$ are of type (iii) with $f : C_i \to p^{-1}(a_i)$ the d_i-fold coverings. Then $[f(C)] = \gamma$ is equivalent to

$$\gamma = \sum_{i=1}^k \iota_*\beta_i + \sum_{i=k+1}^r \iota^\infty_*\beta_i + \sum_{i=r+1}^s d_i P = \sum_{i=1}^r \iota_*\beta_i + \Big(\sum_{i=r+1}^s d_i - \sum_{i=k+1}^r \int_{\beta_i} c_1(K_X)\Big) P \ .$$

Now take $\gamma = \iota_*\beta$ with $\beta \in H_2(X,\mathbb{Z})$ satisfying $\int_\beta c_1(K_X) < 0$ and solve the above equation. The assumption that X is nef implies that the coefficient of P in the last line is always nonnegative. Therefore it is zero if and only if there is no irreducible components of type

(iii) and $\int_{\beta_i} c_1(K_X) = 0$ for those of type (ii). Then connectedness of the domain curve C implies either $f(C) \subset \iota(X)$ or $f(C) \subset \iota^\infty(X)$. For the latter case, $\int_{[f(C)]} c_1(K_X) = 0$ and this contradicts the assumption $\int_\beta c_1(K_X) < 0$. Thus $f(C) \subset \iota(X)$. □

Proof. (of Proposition 2.2.) By Lemma 2.3, the $\mathbb{C}^*$-fixed point set is isomorphic to $\bar{M}_{g,0}(X,\beta)$. Then, by the virtual localization [11],

$$N_{g,\iota_*\beta}(\mathbb{P}(K_X \oplus \mathcal{O}_X)) = \int_{[\bar{M}_{g,0}(X,\beta)]^{vir}} e_{\mathbb{C}^*}(R^1\pi_*\mu^*K_X).$$

Here $e_{\mathbb{C}^*}$ is the equivariant Euler class. (In the equation below [11, (24)], the nontrivial contribution comes only from the factor $e(B_5^m)$; $e(B_2^m)$ does not contribute because $\int_\beta c_1(K_X) < 0$.) Since the LHS is independent of the weight, so is the RHS and we can replace it with the nonequivariant integral. □

Proposition 2.4. *Let X be a nef surface and X' be a nef surface which is deformation equivalent to X. Let $\beta \in H_2(X,\mathbb{Z})$ be a class satisfying $\int_\beta c_1(K_X) < 0$ and $\beta' \in H_2(X',\mathbb{Z})$ be the class corresponding to β under a deformation. Then $N_{g,\beta}(K_X) = N_{g,\beta'}(K_{X'})$ for $g \in \mathbb{Z}_{\geq 0}$.*

Proof. Since X and X' are deformation equivalent, $\mathbb{P}(K_X \oplus \mathcal{O}_X)$ and $\mathbb{P}(K_{X'} \oplus \mathcal{O}_{X'})$ are also deformation equivalent. Let $\iota : X \hookrightarrow \mathbb{P}(K_X \oplus \mathcal{O}_X)$ and $\iota' : X' \hookrightarrow \mathbb{P}(K_{X'} \oplus \mathcal{O}_{X'})$ be the inclusions as the zero sections of K_X and $K_{X'}$ respectively.

We have

$$N_{g,\beta}(K_X) = N_{g,\iota_*\beta}(\mathbb{P}(K_X \oplus \mathcal{O}_X)) = N_{g,\iota'_*\beta'}(\mathbb{P}(K_{X'} \oplus \mathcal{O}_{X'})) = N_{g,\beta'}(K_{X'}).$$

The middle equality follows from the deformation invariance of ordinary GW invariants [23, 29]. The first and the third equalities follow from Proposition 2.2. □

3. Cubic surfaces S_3

Here we summarize some facts on cubic surfaces, see e.g. [12, Ch. V, 4] for details.

Let S_3 be a cubic surface. S_3 is realized as a blowing up $\pi : S_3 \to \mathbb{P}^2$ at six points in a general position. Let $e_1, \cdots, e_6$ be the classes of the exceptional curves of π and l be the class of a line in $\mathbb{P}^2$ pulled back by π. Then $l, e_1, \cdots, e_6$ is a basis of $\mathrm{Pic}(S_3)$. Their intersections are

$$l^2 = 1,\ e_i^2 = -1,\ l.e_i = 0,\ e_i.e_j = 0 \text{ if } i \neq j.$$

Let h be the class of the hyperplane section of $\mathbb{P}^3$. Then we have

$$h = -K_{S_3} = 3l - \sum_{i=1}^{6} e_i\ .$$

It is a classical fact that S_3 contains exactly twenty-seven lines which are given as follows:

$$e_i\ (i = 1, \cdots, 6), \quad l - e_i - e_j\ (1 \leq i < j \leq 6), \quad 2l - \sum_{i \neq j} e_i\ (j = 1, \cdots, 6).$$

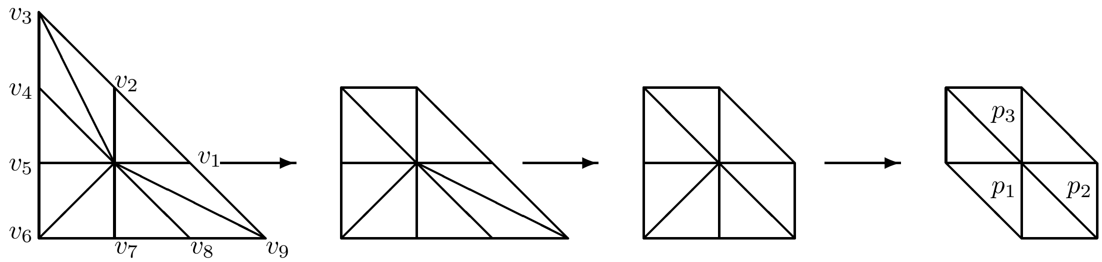

FIGURE 1. $S_3^0 \to S_4^0 \to S_5^0 \to S_6$.

Each one of these is a exceptional curve of the first kind. These twenty-seven lines are the minimal generators of the Mori cone (the cone generated by effective divisors on X modulo numerical equivalence) (cf. [26, (0.6)]).

It is well-known that the Weyl group W_{E_6} of type E_6 acts on $\mathrm{Pic}(S_3)$ as symmetries of configurations of twenty seven lines. Its generators are given as follows.

$$s_i : e_i \leftrightarrow e_{i+1} \ (1 \le i \le 5),$$
$$s_6 : e_1 \mapsto l - e_2 - e_3, \quad e_2 \mapsto l - e_1 - e_3, \quad e_3 \mapsto l - e_1 - e_2, \quad l \mapsto 2l - e_1 - e_2 - e_3.$$

It is known [9, §4] that W_{E_6} coincides with the group of automorphisms of $\mathrm{Pic}(S_3)$ which preserve the intersection form, the canonical class, and the semigroup of effective classes. It is also known that such an automorphism on $\mathrm{Pic}(S_3)$ comes from an isomorphism of S_3.

Hereafter we identify $\mathrm{Pic}(S_3)$ with $H^2(S_3, \mathbb{Z}) \cong H_2(S_3, \mathbb{Z})$. The next lemma was shown in [13, §2.4].

Lemma 3.1. *$N_{g,\beta}(K_{S_3}) = N_{g,w(\beta)}(K_{S_3})$ for $w \in W_{E_6}$.*

Proof. Since the action of w on $H_2(S_3, \mathbb{Z})$ is induced from an isomorphism $\psi : S_3 \to S_3$, we have $N_{g,\beta}(K_{S_3}) = N_{g,\psi_*\beta}(K_{S_3}) = N_{g,w(\beta)}(K_{S_3})$. □

4. NEF TORIC SURFACES DEFORMATION EQUIVALENT TO S_3, S_4, AND S_5

Let S_3^0, S_4^0, and S_5^0 be the nef toric surfaces whose fans are given in Figure 1. Here the nine one-dimensional cones of S_3^0 are generated by

$$v_1 = (1,0), \quad v_2 = (0,1), \quad v_3 = (-1,2), \quad v_4 = (-1,1), \quad v_5 = (-1,0),$$
$$v_6 = (-1,-1), \quad v_7 = (0,-1), \quad v_8 = (1,-1), \quad v_9 = (2,-1).$$

Let the fan of the toric del Pezzo surface S_6 be given in Figure 1 and let p_1, p_2, p_3 be the torus fixed points of S_6 corresponding to the two-dimensional cones generated by $(v_5, v_7), (v_8, v_1), (v_2, v_4)$. S_3^0 (resp. S_4^0, S_5^0) is obtained by blowing up S_6 at p_1, p_2, p_3 (resp. p_1, p_2 and p_1). S_k^0 contains (-2)-curves and its anticanonincal divisor is nef but not ample.

Proposition 4.1. *S_k^0 ($k = 3, 4, 5$) is deformation equivalent to S_k.*

A proof will be given in Appendix A (see Proposition A.2).

Now let us explain the geometry of the nef toric surface S_3^0. The torus-invariant divisors C_i $(1 \le i \le 9)$ corresponding to v_i have the intersections:

$$C_i.C_{i+1} = 1, \quad C_i.C_j = 0 \ (j \ne i, i \pm 1), \quad C_i^2 = \begin{cases} -1 & (i = 3,6,9)\ , \\ -2 & (i = 1,2,4,5,7,8)\ , \end{cases} \tag{4.1}$$

and the canonical divisor $K_{S_3^0}$ is rationally equivalent to $-C_1 - \cdots - C_9$. The Mori cone is generated by $C_1, \ldots, C_9$ [27, Proposition 2.26].

Note that $\mathrm{Pic}(S_3^0) \cong \mathrm{Pic}(S_3)$ and an isomorphism is given by the following.

$$\begin{aligned} &C_1 \mapsto e_2 - e_5, \qquad C_2 \mapsto l - e_2 - e_3 - e_6, \qquad C_3 \mapsto e_6, \\ &C_4 \mapsto e_3 - e_6, \qquad C_5 \mapsto l - e_1 - e_3 - e_4, \qquad C_6 \mapsto e_4, \\ &C_7 \mapsto e_1 - e_4, \qquad C_8 \mapsto l - e_1 - e_2 - e_5, \qquad C_9 \mapsto e_5. \end{aligned} \tag{4.2}$$

This is explained as follows. First, in S_6, we regard the torus-invariant divisors C'_1, C'_4, C'_7 corresponding to v_1, v_4, v_7 as the exceptional curves of blowing up of $\mathbb{P}^2$ and identify them with e_2, e_3, e_1. The torus-invariant divisors C'_2, C'_5, C'_8 corresponding to v_2, v_5, v_8 are identified with the proper transforms $l - e_2 - e_3, l - e_1 - e_3, l - e_1 - e_2$ of lines in $\mathbb{P}^2$. Then in S_3^0, C_3, C_6, C_9 are exceptional curves of the blowup at p_3, p_1, p_2 and we identify them with e_6, e_4, e_5. For $i = 1,2,4,5,7,8$, C_i is the proper transform of C'_i. (This identification can be seen from the construction of a deformation in the proof of Proposition A.2.)

From here on, we identify $\mathrm{Pic}(\mathrm{S}_3^0)$ with $H^2(S_3^0, \mathbb{Z}) \cong H_2(S_3^0, \mathbb{Z})$.

Theorem 4.2. *For $g \in \mathbb{Z}_{\ge 0}$ and $\beta \in H_2(S_3, \mathbb{Z})$ such that $K_{S_3}.\beta < 0$,*

$$N_{g,\beta}(K_{S_3}) = N_{g,\beta'}(K_{S_3^0})\ ,$$

where $\beta' \in H_2(S_3^0, \mathbb{Z})$ is the class corresponding to β by eq. (4.2).

Proof. This follows from Propositions 2.4 and 4.1. □

Remark 4.3. The statements similar to Theorem 4.2 hold for S_4, S_5: local GW invariants of S_4 and S_5 are the same as those of S_4^0 and S_5^0. Their generating functions also have expressions analogous to the formula for S_3 (which will be stated in Theorem 5.2). Local GW invariants of S_4 and S_5 appear among those of S_3 with a natural identification of second homology classes $H_2(S_3, \mathbb{Z}) = H_2(S_4, \mathbb{Z}) \oplus \mathbb{Z}e_6 = H_2(S_5, \mathbb{Z}) \oplus \mathbb{Z}e_5 \oplus \mathbb{Z}e_6$. See [20, §6].

5. Formula for the generating function of local GW invariants of S_3

5.1. First we consider the generating function of local GW invariants of S_3^0 with $\beta \in H_2(S_3^0, \mathbb{Z})$ such that $K_{S_3^0}.\beta < 0$. Take a basis $c_1, \ldots, c_7$ of $H_2(S_3^0, \mathbb{Z})$ and let $X_1, \ldots, X_7$ be associated formal variables. For $\beta = a_1 c_1 + \cdots + a_7 c_7 \in H_2(S_3^0, \mathbb{Z})$, denote $X_1^{a_1} \ldots X_7^{a_7}$ by X^β. We write the generating function as

$$F_{S_3^0} = \sum_{\substack{\beta \in H_2(S_3^0, \mathbb{Z}), \\ K_{S_3^0}.\beta < 0}} \sum_{g \ge 0} N_{g,\beta}(K_{S_3^0}) \lambda^{2g-2} X^\beta.$$

Let $t_i = X^{[C_i]}$ $(1 \le i \le 9)$ and $s_i = C_i^2$ (See (4.1)). Define $Z_{S_3^0}$ by

$$Z_{S_3^0} = \prod_{i=1}^{9} \sum_{\nu^i} ((-1)^{s_i} t_i)^{|\nu^i|} e^{\sqrt{-1}\lambda s_i \frac{\kappa(\nu^i)}{2}} W_{\nu^i,\nu^{i+1}}(e^{\sqrt{-1}\lambda}).$$

Here each ν^i $(1 \le i \le 9)$ runs over the set of partitions and $\nu^{10} = \nu^1$ is assumed. For partitions $\mu = (\mu_1, \mu_2, \ldots)$ and $\nu = (\nu_1, \nu_2, \ldots)$,

$$W_{\mu,\nu}(q) = s_\mu(q^\rho) s_\nu(q^{\mu+\rho}) \in \mathbb{Q}(q^{\frac{1}{2}}), \quad |\mu| = \sum_{i \ge 1} \mu_i, \quad \kappa(\mu) = \sum_{i \ge 1} \mu_i(\mu_i - 2i + 1),$$

where $q^{\mu+\rho} = (q^{\mu_i - i + \frac{1}{2}})_{i \ge 1}$, $q^\rho = (q^{-i+\frac{1}{2}})_{i \ge 1}$ and s_μ denotes the Schur function. Define $Z_{(-2)}(t)$ by

$$Z_{(-2)}(t) = \exp\left[- \sum_{j \ge 1} \frac{1}{j} \Big(2 \sin \frac{j\lambda}{2} \Big)^{-2} t^j \right].$$

Lemma 5.1.

$$\exp\left(F_{S_3^0}\right) = \frac{Z_{S_3^0}}{\prod_{i=1,4,7} Z_{(-2)}(t_i) Z_{(-2)}(t_{i+1}) Z_{(-2)}(t_i t_{i+1})}.$$

Proof. Recall that S_3^0 has a canonical $T = (\mathbb{C}^*)^2$-action determined by its fan. Let $K^T_{S_3^0} = -C_1 - \ldots - C_9$ be an T-invariant divisor. For any $\beta \in H_2(S_3^0, \mathbb{Z})$ and $g \in \mathbb{Z}_{\ge 0}$, define $N^T_{g,\beta}(S_3^0)$ by the following equivariant integral:

$$N^T_{g,\beta}(S_3^0) = \int_{[\bar{M}_{g,0}(S_3^0,\beta)^T]^{vir}} \frac{e_T(R^1\pi_*\mu^* K^T_{S_3^0})}{e_T(R^0\pi_*\mu^* K^T_{S_3^0})} \frac{1}{e_T(Norm)}.$$

Here $\bar{M}_{g,0}(S_3^0, \beta)^T$ is the fixed point set of the induced T-action, e_T denotes the equivariant Euler class and $Norm$ is the virtual normal bundle determined by the obstruction theory [11, eqs. (23)(24)]. Note that $N^T_{g,\beta}(S_3^0) = 0$ if there is no effective divisors of the form $\sum_{1 \le i \le 9} a_i [C_i]$ $(a_i \in \mathbb{Z}_{\ge 0})$ which are rationally equivalent to β because $\bar{M}_{g,0}(S_3^0, \beta)^T$ is empty.

Consider the exponential of the generating function for *all* classes

$$\exp\left[\sum_{\beta \in H_2(S_3^0,\mathbb{Z})} \sum_{g \ge 0} N^T_{g,\beta}(S_3^0) \lambda^{2g-2} X^\beta \right]. \tag{5.1}$$

Carrying out the localization calculation in the same way as [31][4] and using the formula for Hodge integrals [24, Theorem 1], we see that (5.1) is equal to $Z_{S_3^0}$.

Next we have to subtract the contributions coming from classes β which does not satisfy $K_{S_3^0}.\beta < 0$. Note that such effective classes are of the forms $a[C_1] + b[C_2]$, $a[C_4] + b[C_5]$ or $a[C_7] + b[C_8]$ $(a, b \in \mathbb{Z}_{\ge 0})$. Therefore

$$\exp\left[\sum_{\substack{\beta \in H_2(S_3^0,\mathbb{Z}) \\ K_{S_3^0}.\beta \ge 0}} \sum_{g \ge 0} N^T_{g,\beta}(S_3^0) \lambda^{2g-2} X^\beta \right] = \prod_{i=1,4,7} \exp\left[\sum_{a,b \in \mathbb{Z}_{\ge 0}} \sum_{g \ge 0} N^T_{g,a[C_i]+b[C_{i+1}]}(S_3^0) \lambda^{2g-2} t_i^a t_{i+1}^b \right]. \tag{5.2}$$

[4]The contribution to $N^T_{g,\beta}(S_3^0)$ from a fixed locus turns out to be completely the same as [31, eqs. (13)(16)]. Thus the summation over genera, second homology classes and fixed loci proceeds in the same manner.

The $i=1$ factor is easily obtained by setting $t_3=t_4=\cdots=t_9=0$ in (5.1). It is equal to

$$Z_{S_3^0}|_{t_3=t_4=\cdots=t_9=0} = Z_{(-2)}(t_1)Z_{(-2)}(t_2)Z_{(-2)}(t_1t_2)\ .$$

The $i=4,7$ factors are similar. Dividing (5.1) by (5.2), we obtain

$$\exp\left[\sum_{\substack{\beta\in H_2(S_3^0,\mathbb{Z}),\\ K_{S_3^0}.\beta<0}}\sum_{g\geq 0} N^T_{g,\beta}(S_3^0)\lambda^{2g-2}X^\beta\right] = \frac{Z_{S_3^0}}{\prod_{i=1,4,7} Z_{(-2)}(t_i)Z_{(-2)}(t_{i+1})Z_{(-2)}(t_it_{i+1})}.$$

By the virtual localization [11], $N^T_{g,\beta}(S_3^0)=N_{g,\beta}(S_3^0)$ for β such that $K_{S_3^0}.\beta<0$. Thus We complete our proof. $\square$

5.2. Next we study the generating function of local GW invariants of S_3. Let $Q=(Q_1,\dots,Q_6,Q_7)$ be a set of formal variables and denote $Q_1^{a_1}Q_2^{a_2}\dots Q_7^{a_7}$ by Q^β for $\beta = a_1e_1+\cdots+a_6e_6+a_7l\in H_2(S_3,\mathbb{Z})$. Define

$$F_d = \sum_{\substack{\beta\in H_2(S_3,\mathbb{Z}),\\ -K_{S_3}.\beta=d}}\sum_{g\in\mathbb{Z}_{\geq 0}} N_{g,\beta}(K_{S_3})\lambda^{2g-2}Q^\beta\ , \qquad (d\in\mathbb{Z}_{\geq 1}),$$

and $F_{S_3}:=\sum_{d\geq 1}F_d$.

Theorem 5.2. *With the following identification of the parameters*

$$\begin{aligned} &t_1=Q^{e_2-e_5},\quad t_2=Q^{l-e_2-e_3-e_6},\quad t_3=Q^{e_6},\quad t_4=Q^{e_3-e_6},\quad t_5=Q^{l-e_1-e_3-e_4},\\ &t_6=Q^{e_4},\quad t_7=Q^{e_1-e_4},\quad t_8=Q^{l-e_1-e_2-e_5},\quad t_9=Q^{e_5}, \end{aligned} \tag{5.3}$$

we have

$$\exp\left(F_{S_3}\right)=\exp\left(F_{S_3^0}\right).$$

Proof. This follows from Theorem 4.2 and Lemma 5.1. The identification (5.3) is determined by (4.2). $\square$

Remark 5.3. In [8], Diaconescu and Florea obtained a formula for F_{S_3} which is different from ours (eq. (3.14) for $k=5$ in *loc. cit.*). It would be an interesting problem to show that these two formulas are equivalent.

Define $m(\beta)$ for $\beta\in H_2(S_3,\mathbb{Z})$ by

$$m(\beta)=\frac{1}{\#\{w\in W_{E_6}\mid w(\beta)=\beta\}}\sum_{w\in W_{E_6}} Q^{w(\beta)}\ .$$

By Lemma 3.1, F_d should be written in terms of these. F_d up to $d=6$ are shown in Appendix C.

6. Gopakumar–Vafa invariants

Let $n^g_\beta(K_{S_3})$ ($g\in\mathbb{Z}_{\geq 0}$, $\beta\in H_2(S_3,\mathbb{Z})$) be numbers defined by the following :

$$F_{S_3}=\sum_{\beta\in H_2(S_3,\mathbb{Z})}\sum_{g\in\mathbb{Z}_{\geq 0}}\sum_{k\geq 1}\frac{n^g_\beta(K_{S_3})}{k}\Big(2\sin\frac{k\lambda}{2}\Big)^{2g-2}Q^{k\beta}.$$

$n^g_\beta(K_{S_3})$ are called Gopakumar–Vafa invariants [10] . They are listed in Table 1.

d	β	$\#\mathcal{O}(\beta)$	genus	g 0	1	2	3	4	5
1	e_6	27	0	1					
2	$-e_1+l$	27	0	-2					
3	l	72	0	3					
	$-e_1-e_2-e_3-e_4-e_5-e_6+3l$	1	1	27	-4				
4	$-e_1-e_2+2l$	216	0	-4					
	$-e_1-e_2-e_3-e_4-e_5+3l$	27	1	-32	5				
5	$-e_1+2l$	432	0	5					
	$-e_1-e_2-e_3-e_4+3l$	216	1	35	-6				
	$-2e_2-e_2-e_3-e_4-e_5-e_6+4l$	27	2	205	-68	7			
6	$-2e_1-e_2+3l$	432	0	-6					
	$2l$	72	0	-6					
	$-e_1-e_2-e_3+3l$	720	1	-36	7				
	$-2e_1-e_2-e_3-e_4-e_5+4l$	270	2	-198	72	-8			
	$-e_1-e_2-e_3-e_4-e_5-e_6+4l$	72	3	-936	498	-108	9		
	$-2e_1-2e_2-2e_3-2e_4-2e_5-2e_6+6l$	1	4	-3780	2636	-846	141	-10	

TABLE 1. Gopakumar–Vafa invariants $n^g_\beta(K_{S_3})$

Remark 6.1. (a) Gopakumar–Vafa invariants $n^g_\beta(K_{S_3})$ of S_3 are integers. Moreover, for each β, $n^g_\beta(K_{S_3})$ is equal to zero for all but finite g. This follows from the same statement for the toric surface S^0_3 ([28, 19]).

(b) One could observe that $n^g_\beta(K_{S_3})$ in Table 1 are zero if g is larger than the genus $\beta.(\beta+K_{S_3})/2+1$ of a nonsingular curve which belongs to β.

(c) The results are in agreement with previous results in [25, Table 3], [22, Table 1, $n=6$], [4, Table 7, $X_3(1,1,1,1)$] obtained by the B-model calculation of mirror symmetry. Also compare with [15, Table 7].

APPENDIX A. NEF TORIC SURFACES AND THEIR DEFORMATIONS

The following classification is due to Batyrev [3] (see also [4, Table 1]).

Lemma A.1. *There are exactly sixteen nef toric surfaces, whose fans are shown in Figure 2.*

We will refer the nef toric surfaces using the numbers shown in frames in Figure 2.

Proof. The minimal nef toric surfaces are $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$, and the Hirzebruch surface $\mathbb{F}_2$, which are No. 1, No. 2, and No. 4 respectively. Nef toric surfaces are obtained from them by blowing up at a torus-fixed point successively. By the nef condition, we must blow-up at a torus-fixed point which is not on a torus-fixed (-2)-curve. All possible patterns of blowing-ups are listed in Figure 2. Note that No. 13, 15, and 16 can no longer be blown-up to nef toric surfaces, since all of their torus-fixed points are on a torus-fixed (-2)-curve. This completes the classification. □

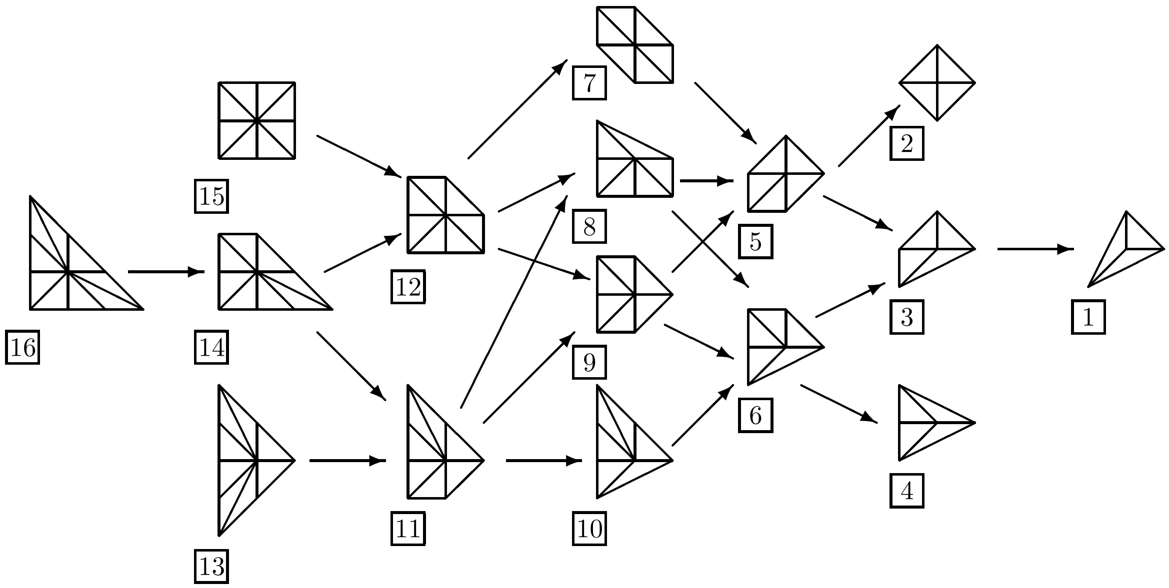

FIGURE 2. Classification of nef toric surfaces. The arrows indicate blow-downs. The numbers in frames are reference numbers. Note that S_3^0, S_4^0, and S_5^0 introduced in §4 are No. 16, 14, and 12, respectively.

Proposition A.2. *A nef toric surface has a smooth versal deformation family of dimension $h^1(\Theta)$, whose general member is a del Pezzo surface of degree c_1^2.*

$h^1(\Theta)$ and c_1^2 are given in Table 2.

Proof. Note that $h^2(\Theta) = 0$ for any smooth compact toric surface (Corollary B.2). This implies smoothness of a versal deformation family [17].

Versal deformation families of nef toric surfaces are constructed inductively as follows. Let $\pi : \tilde{S} \to S$ be one of the blowing-ups in Figure 2. Let $P \in S$ be the center of the blowing-up π which is the intersection of two torus-fixed curves C_1 and C_2 (see Figure 3). By comparing Table 2 with Figure 2, we have

$$h^1(\tilde{S},\Theta) = \begin{cases} h^1(S,\Theta) & \text{if } C_1^2 > -1,\ C_2^2 > -1\ , \\ h^1(S,\Theta)+1 & \text{if } C_1^2 = -1,\ C_2^2 > -1\ , \\ h^1(S,\Theta)+2 & \text{if } C_1^2 = C_2^2 = -1\ . \end{cases} \tag{A.1}$$

Since smooth rational curves on complex surfaces with self-intersection ≥ -1 is stable under small deformations [18, Example in p.86] (see also [2, IV. 3.1]), a complete deformation family of $\tilde{S}$ can be found as a simultaneous blowing-up of a complete deformation family of S. Furthermore, by eq. (A.1), we can find a versal deformation family of $\tilde{S}$ as follows. First, we consider a versal deformation family $\mathcal{S}$ of S on which C_1 and C_2 deform holomorphically. If both of C_1 and C_2 have self-intersection > -1, simultaneous blowing up of $\mathcal{S}$ at P gives a versal deformation family of $\tilde{S}$ which is of dimension $h^1(S,\Theta)$. If $C_1^2 = -1$ and $C_2^2 > -1$, we move the center P in the C_2 direction (see Figure 3) and blow $\mathcal{S}$ up simultaneously to get a versal deformation family of $\tilde{S}$ which is of dimension $h^1(S,\Theta)+1$. If $C_1^2 = C_2^2 = -1$, we move the center P in the whole direction and blow $\mathcal{S}$ up simultaneously to get a versal

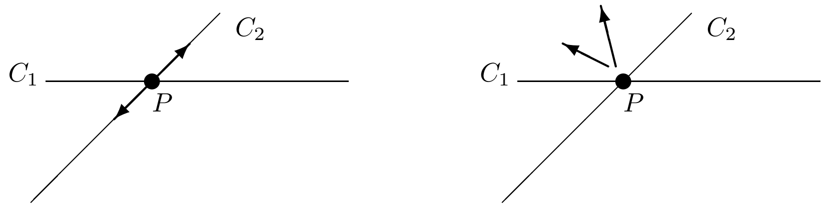

FIGURE 3. The center P of a blowing-up (C_1 and C_2 are torus-fixed curves) and its moving. The left is the case with $C_1^2 = -1$, $C_2^2 \geq 0$ and the right is the case with $C_1^2 = C_2^2 = -1$.

deformation family of $\tilde{S}$ which is of dimension $h^1(S, \Theta) + 2$.
Thus we can find versal deformation families of nef toric surfaces inductively. It is easy to see that their general members are del Pezzo surfaces. $\square$

Deformation type	I	II	III		IV		V				VI		VII			$VIII$
No.	1	3	2	4	5	6	7	8	9	10	11	12	13	14	15	16
c_1^2	9	8	8		7		6				5		4			3
c_2	3	4	4		5		6				7		8			9
$-\frac{7c_1^2-5c_2}{6}$	-8	-6	-6		-4		-2				0		2			4
$h^0(\Theta)$	8	6	6	7	4	5	2	4	3	5	3	2	3	2	2	2
$h^2(\Theta)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$h^1(\Theta)$	0	0	0	1	0	1	0	2	1	3	3	2	5	4	4	6

TABLE 2. Eight deformation types and $h^1(\Theta)\left(= -\frac{7c_1^2-5c_2}{6} + h^0(\Theta) + h^2(\Theta)\right)$.

APPENDIX B. UNOBSTRUCTEDNESS

Let X be a smooth compact toric surface, $D := D_1 + \cdots + D_r$ be the sum of all torus invariant divisors $D_1, \cdots, D_r$, and $\Theta(-\log D)$ be the sheaf of germs of holomorphic vector fields with logarithmic zeros along D.

Lemma B.1. $H^2(X, \Theta(-\log D)) = 0$.

Proof. Since $\Theta(-\log D) = \mathcal{O} \otimes_{\mathbb{Z}} N$ (cf. [27, Proposition 3.1]), where N is the 2-dimensional lattice such that the fan of X sits in $N \otimes \mathbb{R}$. $H^2(X, \Theta(-\log D)) = H^2(X, \mathcal{O} \otimes_{\mathbb{Z}} N) = H^2(X, \mathcal{O} \oplus \mathcal{O}) = 0$, since $H^2(X, \mathcal{O}) = 0$ (cf. [27, Corollary 2.8]). $\square$

Corollary B.2. $H^2(X, \Theta) = 0$.

Proof. From the exact sequence (cf. [27, Theorem 3.12])

$$0 \longrightarrow \Theta(-\log D) \longrightarrow \Theta \longrightarrow \oplus_{i=1}^r \mathcal{O}(D_i)|_{D_i} \longrightarrow 0,$$

and Lemma B.1, we have $H^2(X, \Theta) = 0$. $\square$

Appendix C. F_d $(1 \le d \le 6)$

Let $\mathsf{b[k]} := (2\sin\frac{k\lambda}{2})^2$.

$$F_1 = \frac{1}{\mathsf{b[1]}}\, m(e_6)\ , \qquad F_2 = \frac{1}{2\cdot \mathsf{b[2]}}\, m(2e_6) + \frac{-2}{\mathsf{b[1]}}\, m(-e_1+l)\ ,$$

$$F_3 = \frac{1}{3\cdot\mathsf{b[3]}}\, m(3e_6) + \frac{3}{\mathsf{b[1]}}\, m(l) + \Big(-4 + \frac{27}{\mathsf{b[1]}}\Big)\, m(-e_1-e_2-e_3-e_4-e_5-e_6+3l)\ ,$$

$$\begin{aligned} F_4 = {} & \frac{1}{4\cdot\mathsf{b[4]}}\, m(4e_6) + \frac{-2}{2\cdot\mathsf{b[2]}}\, m(-2e_1+2l) + \frac{-4}{\mathsf{b[1]}}\, m(-e_1-e_2+2l) \\ & + \Big(5 + \frac{-32}{\mathsf{b[1]}}\Big)\, m(-e_1-e_2-e_3-e_4-e_5+3l) \end{aligned}$$

$$\begin{aligned} F_5 = {} & \frac{1}{5\cdot\mathsf{b[5]}}\, m(5e_6) + \frac{5}{\mathsf{b[1]}}\, m(-e_1+2l) + \Big(-6 + \frac{35}{\mathsf{b[1]}}\Big)\, m(-e_1-e_2-e_3-e_4+3l) \\ & + \Big(7\cdot\mathsf{b[1]} - 68 + \frac{205}{\mathsf{b[1]}}\Big)\, m(-2e_1-e_2-e_3-e_4-e_5-e_6+4l)\ , \end{aligned}$$

$$\begin{aligned} F_6 = {} & \frac{1}{6\cdot\mathsf{b[6]}}\, m(6e_6) - \frac{2}{3\cdot\mathsf{b[3]}}\, m(-3e_1+3l) + \Big(\frac{3}{2\cdot\mathsf{b[2]}} - \frac{6}{\mathsf{b[1]}}\Big)\, m(2l) \\ & + \Big(7 - \frac{36}{\mathsf{b[1]}}\Big)\, m(-e_1-e_2-e_3+3l) \\ & + \Big(-8\cdot\mathsf{b[1]} + 72 - \frac{198}{\mathsf{b[1]}}\Big)\, m(-2e_1-e_2-e_3-e_4-e_5+4l) \\ & + \Big(9\cdot\mathsf{b[1]}^2 - 108\cdot\mathsf{b[1]} + 498 - \frac{936}{\mathsf{b[1]}}\Big)\, m(-e_1-e_2-e_3-e_4-e_5-e_6+4l) \\ & + \left(\frac{1}{2}\Big(-4 + \frac{27}{\mathsf{b[2]}}\Big) - 10\cdot\mathsf{b[1]}^3 + 141\cdot\mathsf{b[2]}^2 - 846\cdot\mathsf{b[1]} + 2636 - \frac{3780}{\mathsf{b[1]}}\right) \\ & \times m(-2e_1-2e_2-2e_3-2e_4-2e_5-2e_6+6l)\ . \end{aligned}$$

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